

Harmonic Superspaces in Low Dimensions

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Abstract

Harmonic superspaces for spacetimes of dimension $d \leq 3$ are constructed. Some applications are given.

1 Introduction

It is well known that the spaces which arise in flat space twistor theory, including (complex, compactified) Minkowski space itself, can be constructed as homogeneous spaces of the complexified conformal group, which is $SL(4; \mathbb{C})$ in the four dimensional case. In fact, all of the basic twistor spaces are generalized flag manifolds, which are, by definition, spaces of the form $P \backslash G$ where P is a parabolic subgroup of a complex, simple Lie group, G [1]. At the Lie algebra level, a parabolic subalgebra, \mathfrak{p} , of a Lie algebra, \mathfrak{g} , is one which contains the Borel subalgebra, \mathfrak{b} , the latter consisting of the Cartan subalgebra and the elements of \mathfrak{g} corresponding to the positive roots. Viewed in this light, twistor theory can be seen as a particular application of homogeneous space theory [9].

This point of view has a natural generalization to supersymmetry with the complexified superconformal group as a starting point. The role of flag supermanifolds has been particularly emphasized by Manin [2], although supertwistors were introduced by Ferber [3] and exploited in a field theory context by Witten [4] and others. However, in the supersymmetric case, there are a lot more spaces which one can consider, especially when one has extended supersymmetry. In fact, two main classes of flag supermanifolds can be constructed: those whose body is a conventional twistor space, which we shall refer to as supertwistor spaces, and those whose body is Minkowski space times an internal flag manifold which can be thought of as a homogeneous space of the internal symmetry group part of the superconformal group. This latter type of superspace has been given the name harmonic superspace and was first introduced by GIKOS [5][6] and by Karlhede *et al* [7]. A review of the various superspaces which can be constructed in four dimensions is given in [8].

In this note we construct all possible harmonic superspaces for spacetimes of dimension $d \leq 3$, which can be obtained as generalized flag supermanifolds of the superconformal group. Although there are only a few ordinary twistor spaces in $d=3$ and none in $d=1$ or 2 one can still construct a large number of harmonic superspaces. After giving a brief review of the general twistor approach, emphasizing the rôle of double fibrations, we construct the various harmonic superspaces which can arise. In the applications to field theory one is mainly interested in real, non-compact Minkowski superspace whereas the corresponding complex flag supermanifold has a compact body. The non-

compact Minkowski superspace can be viewed as an open set in this space, and when reality is imposed the harmonic double fibration is replaced by a single fibration of harmonic superspace over ordinary Minkowski superspace. One then finds that harmonic superspace, although not a complex space, is a CR supermanifold. We conclude with a few simple applications.

2 General Framework

Let X , Y and Z be complex (super)manifolds and suppose Y fibres over X and over Z . Then we have a double fibration given by the following diagram:

$$\begin{array}{ccc}
 & Y & \\
 \pi_1 \swarrow & & \searrow \pi_2 \\
 Z & \Longleftrightarrow & X
 \end{array}$$

(1)

Information on X can be transferred to Z via the correspondence space Y and vice-versa. In the applications X will be identified with Minkowski (super)space so physics can be translated to information on Z . If $x \in X$ then write $\hat{x} = \pi_1 \pi_2^{-1}(x) \subset Z$ for the corresponding subspace in Z . If $z \in Z$ then write $\tilde{z} = \pi_2 \pi_1^{-1}(z) \subset X$. To ensure that a subspace $\hat{x} \subset Z$ corresponds to only one point in X and that $\tilde{z} \subset X$ corresponds to only one point in Z we demand that π_1 is 1-1 on the fibres of π_2 and π_2 is 1-1 on the fibres of π_1 . (This is equivalent to demanding that $(\pi_1, \pi_2) : Y \rightarrow (Z, X)$ is an embedding).

A class of double fibrations can be constructed using homogeneous spaces of a complex semi-simple Lie Group or a complex, basic, classical, simple Lie supergroup, G . In fact, if P and P' are parabolic subgroups of G then

$P \cap P'$ is also a parabolic subgroup of G and the following diagram can be constructed [9]:

$$\begin{array}{ccc}
 & P \cap P' \backslash G & \\
 \pi_1 \swarrow & & \searrow \pi_2 \\
 P \backslash G & \Longleftrightarrow & P' \backslash G
 \end{array}$$

(2)

$P \cap P' \backslash G$ has typical fibre $P \cap P' \backslash P$ over $P \backslash G$ and typical fibre $P \cap P' \backslash P'$ over $P' \backslash G$. We will be interested in G being the superconformal group.

Parabolic subgroups of a complex, semi-simple Lie group, G , are constructed as follows (see e.g. [9]):

Suppose G has Lie algebra \mathfrak{g} . Given a Cartan subalgebra, $\mathfrak{h} \subset \mathfrak{g}$, and a set of positive roots, $\Delta^+(\mathfrak{g})$, of \mathfrak{g} , one has a Borel subalgebra, $\mathfrak{b} \subset \mathfrak{g}$, given by the direct sum of \mathfrak{h} and the positive root spaces:

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+(\mathfrak{g})} \mathfrak{g}_\alpha \quad (3)$$

where \mathfrak{g}_α is the root space corresponding to the root α .

Let \mathfrak{g} have the simple positive roots

$$S = \{\alpha_1, \dots, \alpha_n\}. \quad (4)$$

Take a subset $S_{\mathfrak{p}} \subset S$ and denote by \mathfrak{l} the set of root spaces spanned by $S_{\mathfrak{p}}$. Now let \mathfrak{p} be the direct sum of \mathfrak{b} and the root spaces corresponding to the negative roots, $\Delta^-(\mathfrak{l})$, in \mathfrak{l} :

$$\mathfrak{p} = \mathfrak{b} \oplus \bigoplus_{\alpha \in \Delta^-(\mathfrak{l})} \mathfrak{g}_\alpha. \quad (5)$$

\mathfrak{p} contains \mathfrak{b} therefore it is a parabolic subalgebra of \mathfrak{g} (by definition). Corresponding to \mathfrak{p} there is a parabolic subgroup $P \subset G$.

The homogeneous space $P \backslash G$ is called a **generalized flag manifold**. Suppose $S \backslash S_{\mathfrak{p}} = \{\alpha_{i_1}, \dots, \alpha_{i_k}\}, i_1 \leq \dots \leq i_k$ then label the corresponding parabolic subgroup $P_{i_1 \dots i_k}$ and the flag $F_{i_1 \dots i_k}$.

In the supersymmetric case we will be dealing with a complex Lie supergroup, G , with basic, classical, simple Lie superalgebra, \mathfrak{g} . The construction of parabolic subgroups is similar to the construction described above. The spaces $P \backslash G$ are called **generalized flag supermanifolds**. Here we have a number of (Weyl) inequivalent Borel subalgebras to choose from. This leads to a slightly different notation. If $\alpha_{i_1|j_1}, \dots, \alpha_{i_n|j_n}$ (where $i_l + j_l = l$) are the simple roots and $S \backslash S_{\mathfrak{p}} = \{\alpha_{i_1|j_1}, \dots, \alpha_{i_k|j_k}\}, i_1 \leq \dots \leq i_k, j_1 \leq \dots \leq j_k$ then label the corresponding parabolic subgroup $P_{i_1|j_1 \dots i_k|j_k}$ and the flag $F_{i_1|j_1 \dots i_k|j_k}$.

3 Superconformal Groups in Low Dimensions

The complex conformal groups for $d=1,2,3$ are $Sp(1)$, $Sp(1) \times Sp(1)$ and $Sp(2)$ respectively. $Sp(1)$ has only one root, $S = \{\alpha_1\}$. Hence there is only one flag manifold for $d=1$, F_1 :

$$F_1 = P_1 \backslash Sp(1) = \mathbb{CP}^1 \quad (6)$$

where

$$P_1 = \left\{ \begin{pmatrix} \times & \\ \times & \times \end{pmatrix} \in Sp(1) \right\}. \quad (7)$$

We identify complex 1d Minkowski space, \mathbb{M} , via the map $\mathbb{C} \longrightarrow Sp(1)$ given by

$$x \longmapsto s(x) := \begin{pmatrix} 1 & ix \\ 0 & 1 \end{pmatrix}. \quad (8)$$

If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(1)$ then the right action of $Sp(1)$ on \mathbb{M} is easily found to be (where defined, in the non-compact case)

$$x \longrightarrow x' = \frac{b + idx}{a + ixc}. \quad (9)$$

Reality is obtained by demanding that $s^\dagger K_2 s = K_2$ where $K_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; in this case $x = x^\dagger$.

For d=2 compact, complex Minkowski space is $F_1 \times F_1$. Non-compact complex 2d Minkowski space, \mathbb{M} , can be identified in $F_1 \times F_1$ via the map $\mathbb{C}^2 \longrightarrow Sp(1) \times Sp(1)$ given by

$$(x^+ = x^0 + x^1, x^- = x^0 - x^1) \longmapsto s(x) := \left(\begin{pmatrix} 1 & ix^+ \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & ix^- \\ 0 & 1 \end{pmatrix} \right). \quad (10)$$

The right action of $Sp(1) \times Sp(1)$ on $s(x)$ is the action of the conformal transformations. Hence $Sp(1) \times Sp(1)$ is the conformal group for d=2. Reality is obtained as in the d=1 case.

For d=3, $Sp(2)$ is the relevant group. There are two simple roots, $S = \{\alpha_1, \alpha_2\}$. Consider the flag F_2 :

$$F_2 = P_2 \backslash Sp(2) \quad (11)$$

where

$$P_2 = \left\{ \begin{pmatrix} \times & \times & & \\ \times & \times & & \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix} \in Sp(2) \right\}. \quad (12)$$

Complex 3d Minkowski space, \mathbb{M} , can be identified with an open chart for F_2 via the map $\mathbb{C}^3 \longrightarrow Sp(2)$ given by

$$x \longmapsto s(x) := \begin{pmatrix} 1 & ix^{\alpha\beta} \\ 0 & 1 \end{pmatrix} \quad (13)$$

where $x^{\alpha\beta} = x^{\beta\alpha}$, $\alpha, \beta = 1, 2$.

The conformal transformations are given by the right action of $Sp(2)$ on $s(x)$. Reality is obtained by imposing $s^\dagger K_4 s = K_4$ where $K_4 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}$.

The superconformal groups are $OSp(N|1)$, $OSp(M|1) \times OSp(N|1)$ and $OSp(N|2)$ for $d=1,2,3$ respectively. The Lie supergroup $OSp(N|M)$ is given by

$$OSp(N|M) = \{g \in GL(N|M) \mid g^{ST} J g = J\} \quad (14)$$

where $J = \left(\begin{array}{c|c} S_M & 0 \\ \hline 0 & K_N \end{array} \right)$ and

$$S_M = \begin{pmatrix} 0 & 1_M \\ -1_M & 0 \end{pmatrix}, \quad K_N = \begin{cases} \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} & \text{if } N = 2n \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1_n \\ 0 & 1_n & 0 \end{pmatrix} & \text{if } N = 2n + 1 \end{cases}. \quad (15)$$

We will use the system of simple roots for $OSp(N|M)$ given by

$$S = \{\alpha_{1|0}, \dots, \alpha_{M|0}, \alpha_{M|1}, \dots, \alpha_{M|n}\} \quad (16)$$

For $OSp(N|1)$ we have $S = \{\alpha_{1|0}, \alpha_{1|1}, \dots, \alpha_{1|n}\}$. Consider the flag $F_{1|1}$.

$$F_{1|1} = P_{1|1} \backslash OSp(N|1) \quad (17)$$

where

$$P_{1|1} = \left\{ \left(\begin{array}{c|ccc} \times & & & & \\ \times & \times & \times & \dots & \times \\ \hline \times & & \times & \dots & \times \\ \vdots & & \vdots & & \vdots \\ \times & & \times & \dots & \times \end{array} \right) \in OSp(N|1) \right\}. \quad (18)$$

1d Minkowski superspace has coordinates (x, θ^i) , $i = 1, \dots, N$ and it can be identified in $F_{1|1}$ via the map $\mathbb{C}^{1|N} \longrightarrow OSp(N|1)$ given by

$$(x, \theta^i) \longmapsto s(x, \theta) := \left(\begin{array}{cc|c} 1 & ix & -\theta^i \\ 0 & 1 & 0 \\ \hline 0 & \theta_i & 1_N \end{array} \right). \quad (19)$$

where the $O(N)$ indices are raised and lowered by K_N , i.e. $\theta^i = K_N^{ij} \theta_j$.

The right action of $OSp(N|1)$ on $s(x, \theta)$ is the action of the superconformal transformations. For the real case we impose $s^\dagger J' s = J'$ where

$J' = \left(\begin{array}{c|c} K_2 & 0 \\ \hline 0 & 1_N \end{array} \right)$. Then $x = x^\dagger$ and $\theta_i = \theta^{i\dagger}$.

For d=2, Minkowski superspace can be identified in $F(OSp(M|1))_{1|1} \times F(OSp(N|1))_{1|1}$ via the map $\mathbb{C}^{2|M+N} \longrightarrow OSp(M|1) \times OSp(N|1)$ given by

$$(x, \theta) \longmapsto s(x, \theta) := \left(\left(\begin{array}{cc|c} 1 & ix^+ & -\theta^{+i} \\ 0 & 1 & 0 \\ \hline 0 & \theta_i^+ & 1_M \end{array} \right), \left(\begin{array}{cc|c} 1 & ix^- & -\theta^{-i} \\ 0 & 1 & 0 \\ \hline 0 & \theta_i^- & 1_N \end{array} \right) \right). \quad (20)$$

In the real case $x^\pm = x^{\pm\dagger}$, $\theta_i^\pm = \theta^{\pm i\dagger}$.

$OSp(N|2)$ has simple roots $S = \{\alpha_{1|0}, \alpha_{2|0}, \alpha_{2|1}, \dots, \alpha_{2|N}\}$. Elements of the parabolic $P_{2|0}$ have the form

$$\left(\begin{array}{cccc|ccc} \times & \times & & & & & & \\ \times & \times & & & & & & \\ \times & \times & \times & \times & \times & \dots & \times & \\ \times & \times & \times & \times & \times & \dots & \times & \\ \hline \times & \times & & & \times & \dots & \times & \\ \vdots & \vdots & & & \vdots & & \vdots & \\ \times & \times & & & \times & \dots & \times & \end{array} \right). \quad (21)$$

Minkowski superspace is identified in $F_{2|0}$ via the map $\mathbb{C}^{3|2N} \longrightarrow OSp(N|2)$ given by

$$(x^{\alpha\beta}, \theta^{\alpha i}) \longmapsto s(x, \theta) := \left(\begin{array}{cc|c} 1 & ix^{\alpha\beta} & -\theta^{\alpha i} \\ 0 & 1 & 0 \\ \hline 0 & \theta_i^\beta & 1_N \end{array} \right). \quad (22)$$

To get reality we demand that $s^\dagger J'' s = J''$ where $J'' = \left(\begin{array}{c|c} K_4 & 0 \\ \hline 0 & 1_N \end{array} \right)$. Then $x^{\alpha\beta} = x^{\alpha\beta\dagger}$ and $\theta_i^\alpha = \theta^{\alpha i\dagger}$.

4 Harmonic superspaces

Here we classify all possible harmonic superspaces for d=1,2,3. They are the product of Minkowski superspace with any flag manifold of the internal symmetry group $O(N)$ (or $O(M) \times O(N)$ for d=2).

d=3: In the notation of Section 2, if $G = OSp(N|2)$ and $P' = P_{2|0}$ then we have the double fibration

$$\begin{array}{ccc}
 & P \cap P_{2|0} \backslash OSp(N|2) & \\
 \pi_1 \swarrow & & \searrow \pi_2 \\
 P \backslash OSp(N|2) & \Longleftrightarrow & P_{2|0} \backslash OSp(N|2)
 \end{array}
 \tag{23}$$

We restrict $P_{2|0} \backslash OSp(N|2)$ to the open chart that is identified with complex Minkowski superspace, \mathbb{M} . Now define

$$\mathbb{M}_H = \pi_2^{-1}(\mathbb{M}) \subset P \cap P_{2|0} \backslash OSp(N|2), \tag{24}$$

$$\mathbb{M}_A = \pi_1 \pi_2^{-1}(\mathbb{M}) \subset P \backslash OSp(N|2). \tag{25}$$

\mathbb{M}_H is called **harmonic superspace** and \mathbb{M}_A is called **analytic superspace**. Under this restriction the diagram becomes

$$\begin{array}{ccc}
 & \mathbb{M}_H & \\
 \pi_1 \swarrow & & \searrow \pi_2 \\
 \mathbb{M}_A & \Longleftrightarrow & \mathbb{M}
 \end{array}
 \tag{26}$$

The essential difference between the supertwistor spaces and the harmonic superspaces is that for the former the deviation of the body of Z (in diagram 2) from the body of \mathbb{M} lies entirely in the spacetime sector (the Sp part of OSp) whereas in the harmonic case it lies entirely in the internal sector (the $O(N)$ part of OSp).

P is chosen so that its even part differs from the even part of $P_{2|0}$ only in the internal sector. This amounts to choosing a parabolic subgroup of $O(N)$. $O(N)$ has n simple roots (where $N = 2n$ ($2n + 1$) if N is even (odd)):

$$S = \{\alpha_1, \dots, \alpha_n\} \quad (27)$$

The flag manifold $F_{i_1 \dots i_k}$ has the real representation

$$F_{i_1 \dots i_k} = \frac{O(N)}{O(N - 2i_k) \times U(i_k - i_{k-1}) \times \dots \times U(i_2 - i_1) \times U(i_1)}. \quad (28)$$

(Real representation: $g \in O(N)$ satisfies $g^\dagger g = 1$).

This is true for $N \neq 4$. When $N=4$ one finds that the previous equation is correct for F_{12} and F_2 but $F_1 = \frac{O(4)}{U(2)}$.

If $P_{i_1 \dots i_k}$ is the chosen parabolic of $O(N)$ then the corresponding parabolic of $OSp(N|2)$ is $P = P_{2|i_1 \dots 2|i_k}$, then $P \cap P' = P_{2|02|i_1 \dots 2|i_k}$. The choice of parabolic affects the dimension of the odd part of analytic superspace. \mathbb{M} has $2N$ anticommuting coordinates but \mathbb{M}_A has only $2(N - i_1)$. (c.f. the $d=4, N=1$ case where chiral superspace has half as many coordinates as ordinary superspace). For example, in $N = 3$, there is one possible internal flag F_1 which can be presented as $U(1) \backslash SU(2)$. This case has been studied in [12].

\mathbb{M}_H has fibre $F_{i_1 \dots i_k}$ over \mathbb{M} and as $\mathbb{M}_H = \pi_2^{-1}(\mathbb{M})$, $\mathbb{M}_H = \mathbb{M} \times F_{i_1 \dots i_k}$. Hence coordinates for \mathbb{M}_H are $(x^{\alpha\beta}, \theta^{\alpha i}, u_I^i)$ where u_I^i are homogeneous coordinates for $F_{i_1 \dots i_k}$ (I is the isotropy, i.e. parabolic, group index). \mathbb{M}_H can be identified in $F_{2|02|i_1 \dots 2|i_k}$ via the map $\mathbb{M}_H \longrightarrow OSp(N|2)$ given by

$$(x^{\alpha\beta}, \theta^{\alpha i}, u_I^i) \longmapsto s(x, \theta, u) := \left(\begin{array}{cc|c} 1 & ix^{\alpha\beta} & -\theta^{\alpha i} \\ 0 & 1 & 0 \\ \hline 0 & \hat{\theta}_I^\alpha & u_I^i \end{array} \right) \quad (29)$$

where $\hat{\theta}_I^\alpha = u_I^i \theta_i^\alpha$.

There is a natural projection from \mathbb{M}_H to \mathbb{M} given by $(x^{\alpha\beta}, \theta^{\alpha i}, u_I^i) \longmapsto (x^{\alpha\beta}, \theta^{\alpha i})$. The fibre over \mathbb{M}_A has dimension $0|2i_1$. Let coordinates for \mathbb{M}_A

be

$$z = (x_A^{\alpha\beta}, \hat{\theta}^{\alpha I}, (I \neq 1, \dots, i_1), v_I^i). \quad (30)$$

The projection from \mathbb{M}_H to \mathbb{M}_A is given, in local coordinates, by

$$x_A^{\alpha\beta} = x^{\alpha\beta} + i \sum_{I=1}^{i_1} \hat{\theta}^{(\alpha I} \hat{\theta}_I^{\beta)}, \quad \hat{\theta}^{\alpha I} = \theta^{\alpha i} u_i^I, \quad (u_i^I = u^{-1}), \quad v_I^i = u_I^i. \quad (31)$$

The double fibration is

$$\begin{array}{ccc} & (x^{\alpha\beta}, \theta^{\alpha i}, u_I^i) & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ (x_A^{\alpha\beta}, \hat{\theta}^{\alpha I_1+1}, \dots, \hat{\theta}^{\alpha N}, u_I^i) & \Longleftrightarrow & (x^{\alpha\beta}, \theta^{\alpha i}) \end{array} \quad (32)$$

We have covariant derivatives
on \mathbb{M} :

$$D_{\alpha\beta} = \partial_{\alpha\beta}, \quad D_{\alpha i} = \partial_{\alpha i} - i\theta_i^\beta \partial_{\alpha\beta}; \quad (33)$$

on \mathbb{M}_H :

$$D_{\alpha\beta} = \partial_{\alpha\beta}, \quad \hat{D}_{\alpha I} = u_I^i D_{\alpha i} \quad (34)$$

and the $F_{i_1 \dots i_k}$ coset derivatives;

on \mathbb{M}_A :

$$D_{\alpha\beta} = \partial_{\alpha\beta}, \quad \hat{D}_{\alpha I} = u_I^i D_{\alpha i}, \quad (I = i_1 + 1, \dots, n) \quad (35)$$

and the $F_{i_1 \dots i_k}$ coset derivatives.

Note that the coordinates, z , on \mathbb{M}_A regarded as functions on \mathbb{M}_H satisfy

$$\hat{D}_{\alpha I} z = 0, \quad I = 1, \dots, i_1. \quad (36)$$

Any function, ϕ , satisfying $\hat{D}_{\alpha I} \phi = 0$, $I = 1, \dots, i_1$ is called analytic.

d=1: The 1-dimensional case is very similar to the 3d case. The superconformal group is $OSp(N|1)$ and the relevant parabolics are $P = P_{1|i_1 \dots 1|i_k}$, $P' = P_{1|0}$ and $P \cap P' = P_{1|01|i_1 \dots i_k}$. The double fibration of harmonic superspace over analytic superspace and Minkowski superspace, written in local coordinates, is

$$\begin{array}{ccc}
 & (x, \theta^i, u_I{}^i) & \\
 \pi_1 \swarrow & & \searrow \pi_2 \\
 (x_A, \hat{\theta}^{I_1+1}, \dots, \hat{\theta}^N, u_I{}^i) & \Longleftrightarrow & (x, \theta^i)
 \end{array} \tag{37}$$

where

$$x_A = x + i \sum_{I=1}^{i_1} \hat{\theta}^I \hat{\theta}_I. \tag{38}$$

d=2: The 2d case is different in that the coordinates split into 2 sets of coordinates, each transforming independently under the superconformal group, $G = OSp(M|1) \times OSp(N|1)$. Hence the complex superspaces are products of two flags. The double fibration in the compact case is

$$\begin{array}{ccc}
 & F_{1|01|i_1 \dots 1|i_k} \times F_{1|01|j_1 \dots 1|j_l} & \\
 \pi_1 \swarrow & & \searrow \pi_2 \\
 F_{1|i_1 \dots 1|i_k} \times F_{1|j_1 \dots 1|j_l} & \Longleftrightarrow & F_{1|0} \times F_{1|0}
 \end{array} \tag{39}$$

For example, consider the case where $M = N = 4$, $k = l = 1$, $i_1 = j_1 = 2$, so that analytic superspace lies in $F_{1|2} \times F_{1|2}$ and harmonic superspace lies in $F_{1|01|2} \times F_{1|01|2}$. It appears that these superspaces are related to the superfields recently proposed for $d = 2, N = 4$ supersymmetry by Lindström *et al* [10]. For details see [13].

5 Applications of Harmonic Superspace to d=3 Super Yang-Mills

Suppose we want the fibre of \mathbb{M}_H over \mathbb{M}_A to have dimension $0|2k$. This can be achieved by using a flag $F_{i_1 \dots i_l}$ of $O(N)$ which has $i_1 = k$. Also it is easier to work with spaces which have as few harmonic coordinates as possible, so we require that the flag is minimal, i.e. F has only one subscript. Hence the relevant flags of $O(N)$ are F_k , $1 \leq k \leq n$.

An important theorem in twistor theory, the Ward Observation [1], applied to the double fibration in diagram 2 which relates holomorphic vector bundles on Z to gauge fields on X can also be applied in the harmonic case. It states:

There is a 1-1 correspondence between holomorphic vector bundles on \mathbb{M}_A trivial on the fibres of π_2 and gauge fields on \mathbb{M} with vanishing curvature on $\tilde{z} \subset \mathbb{M} \forall z \in \mathbb{M}_A$.

Given a point $z \in \mathbb{M}_A$, translations in the corresponding subset $\tilde{z} \subset \mathbb{M}$ are generated by $\hat{\nabla}_{\alpha I} = u_I^i \nabla_{\alpha i}$, $I = 1, \dots, k$ (∇ is some connection on \mathbb{M}). Hence vanishing curvature on all $\tilde{z} \subset \mathbb{M}$ means

$$u_I^i u_J^j \{\nabla_{\alpha i}, \nabla_{\beta j}\} = u_I^i u_J^j F_{\alpha i \beta j} = 0. \quad (40)$$

This holds for all u_I^i . If $k = 1$ we have

$$u_1^{(i} u_1^{j)} F_{\alpha i \beta j} = u_1^i u_1^j F_{\alpha(i \beta j)} = 0 \quad (41)$$

and so the symmetric part of the curvature, $F_{\alpha(i \beta j)}$, vanishes. However, if $k \geq 2$ then the curvature vanishes altogether,

$$F_{\alpha i \beta j} = 0. \quad (42)$$

(Actually, $u_I{}^i u_J{}^j K_{ij} = 0$ as $1 \leq I, J \leq n$, so $F_{\alpha i \beta j} = K_{ij} F_{\alpha \beta}$ but $F_{\alpha \beta}$ can be chosen to be 0. It is just a matter of conventionally choosing the potential $A_{\alpha \beta}$ in terms of $A_{\alpha i}$).

So we have two situations arising. Either $F_{\alpha(i\beta j)} = 0$ which is a curvature constraint that leads to an off-shell Yang-Mills multiplet, or $F_{\alpha i \beta j} = 0$ which is the constraint for Chern-Simons theory. The Ward observation tells us that holomorphic vector bundles on \mathbb{M}_A trivial on the fibres of π_2 correspond to gauge fields on \mathbb{M} satisfying either $F_{\alpha(i\beta j)} = 0$ or $F_{\alpha i \beta j} = 0$ depending on which harmonic superspace we have.

So far we have been dealing with complex superspaces. One is really interested in real Minkowski superspace, \mathbb{M} , which is obtained by imposing $x^{\alpha\beta} = x^{\alpha\beta\dagger}$, $\theta_i^\alpha = \theta^{\alpha i\dagger}$ in \mathbb{M} . Under this restriction we find that π_1 becomes 1-1 onto its image and the double fibration collapses to a single fibration. It is now convenient to work with the fibres of π_2 (i.e. the relevant flag of $O(N)$) in the real representation. We have

$$\begin{array}{c}
 M_H = \pi_2^{-1}(M) = M \times \frac{O(N)}{O(N-2k) \times U(k)} \\
 \downarrow \pi \\
 M
 \end{array}
 \tag{43}$$

M_H is a CR supermanifold, i.e. a supermanifold with a CR structure. (The corresponding observation in d=4 was made by Rosly and Schwarz [11]).

We recall that a **CR structure** on a real supermanifold, N , of dimension $2n + m|2n' + m'$ is a complex rank $n|n'$ sub-bundle, F , of the complexified tangent bundle, $T_c N = \mathbb{C} \times N$, which is involutive (i.e. if $X, Y \in \Gamma(F)$ (sections of F) then $[X, Y] \in \Gamma(F)$).

The Ward observation now becomes:

There is a 1-1 correspondence between CR vector bundles on M_H (i.e. the

transition functions H satisfy $X(H) = 0 \forall X \in \Gamma(\overline{F})$) trivial on the $\frac{O(N)}{O(N-2k) \times U(k)}$ fibres and gauge fields on M satisfying $F_{\alpha(i\beta j)} = 0$ (if $k = 1$) or $F_{\alpha i \beta j} = 0$ (if $k > 1$).

In the $k = 1$ case the resulting off-shell multiplet can be used to construct a Yang-Mills action for sufficiently low N . It can also be used to write down an Abelian Chern-Simons action for all N . The $k \geq 2$ case is relevant only for Chern-Simons theory. In some cases we can go off-shell using the $d=4$, $N=3$ Yang-Mills construction of GIKOS [6]. For example consider $N = 6$. Taking $k = 3$ we have the internal flag space $F_3 = \frac{O(6)}{U(3)}$, which is a three dimensional complex space. By relaxing the constraint on the internal space curvature we find the off-shell theory, which involves an infinite number of auxiliary fields due to the dependence on the harmonic coordinates. The action is then the Chern-Simons density for the internal curvature integrated over analytic superspace. Similarly, the $N = 4$ and 5 cases can be taken off-shell. For these theories the relevant internal spaces are $\frac{O(4)}{U(2)}$ and $\frac{O(5)}{U(2)}$ respectively. One encounters a problem in writing down actions when $N > 6$. This is because any parabolic can, at most, reduce the number of odd coordinates in analytic superspace by n ($N = 2n$ or $2n + 1$). Hence the dimension of the measure in the action is at least $(n - 3)$ (n for $d^{2n}\theta$ and -3 for d^3x). Therefore when $n > 3$ one requires that some of the fields in the integrand have negative dimension.

Further details and applications will be given elsewhere [14].

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